BAULKHAM HILLS HS PERSEVERE

2023	TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 3 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
Total marks: 100	**Section I – 10 marks** (pages 2 – 4) • Attempt Questions 1 – 10 • Allow about 15 minutes for this section. **Section II – 90 marks** (pages 5 – 10) • Attempt Questions 11 – 16 • Allow about 2 hours and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section.

Use the multiple-choice answer page in the writing booklet for Questions 1 – 10.

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1 If $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and B is at (3,4,3), how far is A from the origin?

A. $\sqrt{7}$

B. 3

C. $4\sqrt{3}$

D. 6

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2 Only one of these statements is FALSE. Which statement is FALSE?

A. If x is odd, x^2 is also odd

B. $\exists x \in \mathbb{R}, 3^x + 4^x = 5^x$

C. $5^{2n} - 1$ is divisible by 8 for all positive integers n

D. $\forall x, y \in \mathbb{R}$, if $x^2 - y^2 > 0$ then $x - y > 0$

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3 Which expression is equivalent to $\int 3\sqrt{x} \ln x dx$?

A. $2x\sqrt{x}\left(\ln x - \dfrac{2}{3}\right) + C$

B. $2x\sqrt{x}\left(\ln x + \dfrac{2}{3}\right) + C$

C. $\dfrac{1}{\sqrt{x}}\left(\dfrac{3}{2}\ln x - 1\right) + C$

D. $\dfrac{1}{\sqrt{x}}\left(\dfrac{3}{2}\ln x + 1\right) + C$

4 Which expression is equivalent to $\int \frac{1}{x^2+6x+25}\,dx$?

A. $\frac{1}{4}\tan^{-1}\left(\frac{x+3}{4}\right)+C$

B. $\frac{1}{x+3}\ln\left(x^2+6x+25\right)+C$

C. $\tan^{-1}\left(\frac{x+3}{4}\right)+C$

D. $\ln\left(x^2+6x+25\right)+C$

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5 The velocity of a body moving in a straight line is given by $v=f(x)$ where x metres is the displacement relative to the origin and v is the velocity in metres per second. What is the acceleration of the body in metres per second squared?

A. $f'(x)$

B. $f'(v)$

C. $xf'(x)$

D. $f(x)f'(x)$

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6 The points P, Q and R have position vectors $\overrightarrow{OP}=\begin{pmatrix}4\\2\\-3\end{pmatrix}$, $\overrightarrow{OQ}=\begin{pmatrix}6\\3\\-1\end{pmatrix}$ and $\overrightarrow{OR}=\begin{pmatrix}5\\1\\0\end{pmatrix}$.

What is the size of $\angle QPR$, correct to the nearest degree?

A. 25°

B. 42°

C. 45°

D. 49°

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7 If the cube roots of unity are $1, \omega$ and ω^2, then what are the roots of the equation $(x-1)^3+8=0$?

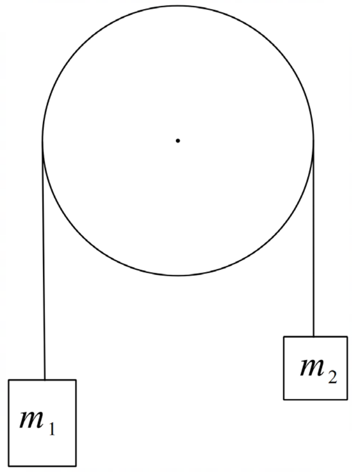
A. $-1,\ -1+2\omega,\ -1-2\omega^2$

B. $-1,\ -1,\ -1$

C. $-1,\ 1-2\omega,\ 1-2\omega^2$

D. $-1,\ 1+2\omega,\ 1+2\omega^2$

8 Two bodies of masses m_1 and m_2 are connected by a light inextensible string and pass over a smooth pulley. If the mass m_1 is coming down, what is the acceleration of the mass m_2?



A. $\dfrac{g(m_1+m_2)}{m_1-m_2}$

B. $\dfrac{g(m_1-m_2)}{m_1+m_2}$

C. $\dfrac{g(m_1m_2)}{m_1-m_2}$

D. $\dfrac{g(m_1m_2)}{m_1+m_2}$

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9 For how many values of x is $\sin x + i\cos 2x$ conjugate to $\cos x - i\sin 2x$?

A. No value of x

B. Only 1 value of x

C. Two values of x

D. An infinite number of values of x

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10 The area of the region enclosed by the curve $z\bar{z} + a(z+\bar{z}) + a = 0$ is 2π square units.

If $a^2 - 7a + 10 = 0$, what is the area, in square units, of the region enclosed by $z\bar{z} + 2a(z+\bar{z}) + a = 0$?

A. 4π

B. 10π

C. 14π

D. 22π

Section II

90 marks

Attempt Questions 11-16

Allow about 2 hours and 45 minutes for this section.

Instructions

- Answer the questions in the appropriate page of your answer booklet.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing paper is available. If you use extra paper, write your name, and your teacher's name on the paper and clearly indicate which question you are answering.

Question 11 (16 marks) Start on the relevant page of your answer booklet

(a) Find $\int \frac{x^2}{x^2+1}\,dx$. **2**

(b) Let $z = \frac{1-7i}{3+4i}$.

(i) Express z in the form $a + ib$. **1**

(ii) Find Arg z^2. **2**

(c) (i) Find real numbers a, b and c such that $\frac{5x^2-4x-9}{(x-2)(x^2-3)} \equiv \frac{a}{x-2} + \frac{bx+c}{x^2-3}$. **2**

(ii) Hence, show that $\int_3^4 \frac{5x^2-4x-9}{(x-2)(x^2-3)}\,dx = \ln\frac{52}{3}$. **2**

(d) It is given that $z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ and $z_2 = 1+i$.

(i) Express $z_1 z_2$ in exponential form. **2**

(ii) Find the smallest positive value of n for which $(z_1 z_2)^n$ is a purely imaginary number. **2**

(e) Consider the statement about a real number x "if $x^5 + 4x^3 + 5x \geq 8x^4 + 2x^2 + 17$ then $x \geq 0$".

(i) Write down the contrapositive of the statement. **1**

(ii) Hence, or otherwise, prove the statement "$\forall x \in \mathbb{R}$, if $x^5 + 4x^3 + 5x \geq 8x^4 + 2x^2 + 17$ then $x \geq 0$". **2**

Question 12 (14 marks) Start on the relevant page of your answer booklet

(a) Use the substitution $t = \tan\frac{x}{2}$ to show that $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{dx}{1+\sin x} = 4 - 2\sqrt{3}$. **3**

(b) (i) On an Argand diagram, shade the region containing all the points representing z such that both $|z-1| \le 1$ and $0 \le \arg z \le \frac{\pi}{6}$. **2**

(ii) Find the area of the shaded region in simplest exact form. **2**

(c) Point A has a position vector of $2\underset{\sim}{i} - 6\underset{\sim}{j} + 5\underset{\sim}{k}$ and point B has a position vector of $\underset{\sim}{i} - 4\underset{\sim}{j} - 3\underset{\sim}{k}$. **2**

The point X lies on the line AB and divides the line in the ratio 3:1.

Find the position vector of X.

(d) If α and β are the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1}\cos\frac{n\pi}{3}$. **3**

(e) The points A, B, C and D form a quadrilateral and represent the complex numbers z_1, z_2, z_3 and z_4 respectively. It is given that $z_1 - z_2 + z_3 - z_4 = 0$. Determine what type of quadrilateral is $ABCD$, giving reasons for your answer. **2**

Question 13 (15 marks) Start on the relevant page of your answer booklet

(a) Find $\displaystyle\int \frac{e^x\,dx}{e^{2x}-1}$. **3**

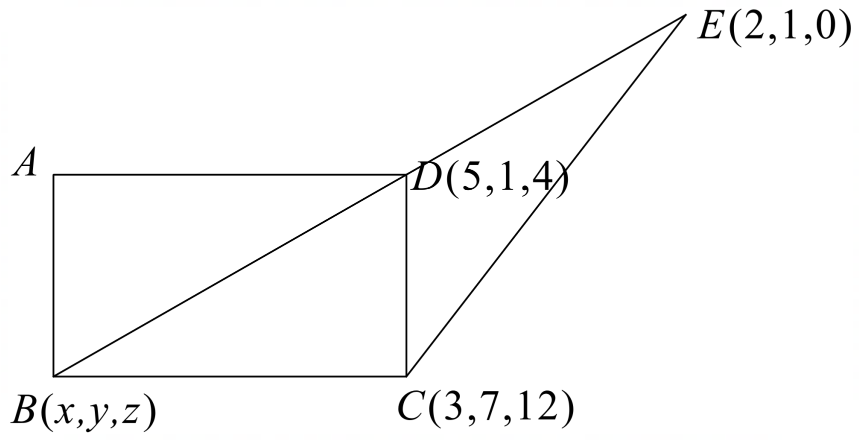
(b) The velocity of a particle moving along the x-axis is given by $v^2 = 12 + 8x - 4x^2$, where x is the displacement of the particle and x is measured in metres.

(i) Prove that the particle moves in simple harmonic motion. **2**

(ii) Find the amplitude, centre of oscillation and the period. **3**

(iii) Initially, the particle is at $x = 3$. Calculate the time taken for the particle to travel a distance of 1 metre from $x = 3$ **and** the velocity when it has travelled 1 metre. **3**

(c) The figure below shows the rectangle $ABCD$, where C has coordinates (3,7,12) and D has coordinates (5,1,4). The point E (2,1,0) is such so that BDE and EC are straight lines.



Use vector methods to determine the coordinates of A. **4**

Question 14 (14 Marks) Start on the relevant page of your answer booklet

(a) Prove, by contradiction, that there are no positive integers p and q such that $4p^2 - q^2 = 25$. **4**

(b) Three numbers a, b and c are all positive.

(i) Use the Arithmetic/Geometric mean inequality, or otherwise, to prove that $(a+b+c)^3 \geq 27abc$. **2**

(ii) Hence, or otherwise, if a, b and c multiply to give 1, prove that $a^5c^4 + b^5a^4 + c^5b^4 \geq 3$. **3**

(c) Let the points $A_1, A_2, A_3, \ldots, A_n$ represent the n-th roots of unity $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$. Suppose also that P represents a complex number z such that $|z| = 1$.

(i) Prove that $\omega_1 + \omega_2 + \omega_3 + \cdots + \omega_n = 0$. **1**

(ii) Show that $|PA_k|^2 = (z - \omega_k)(\overline{z} - \overline{\omega}_k)$ for $k = 1, 2, 3, \ldots, n$. **1**

(iii) Hence prove that $\displaystyle\sum_{k=1}^{n} |PA_k|^2 = 2n$. **3**

Question 15 (17 Marks) Start on the relevant page of your answer booklet

(a) A particle of mass m falls vertically from rest at a height of H metres above the Earth's surface, against a resistive force mkv when its speed is v ms^{-1}. (k is a positive constant). Let x metres be the distance the particle has fallen and g ms^{-2} be the acceleration due to gravity.

(i) Show that the acceleration, a, is given by $a = g - kv$. **1**

(ii) Calculate the terminal velocity of the particle. **1**

(iii) If the particle reaches the Earth's surface with speed V_0, show that **4**

$$\ln\left(1 - \frac{kV_0}{g}\right) + \frac{kV_0}{g} + \frac{k^2 H}{g} = 0.$$

(iv) Show that the time, T, taken to reach the Earth's surface is given by **3**

$$T = \frac{1}{k}\ln\left(\frac{g}{g - kV_0}\right).$$

(v) Show that $V_0 = Tg - kH$. **1**

(vi) Hence, prove that $T < \frac{1}{k} + \frac{kH}{g}$. **1**

(b) A person standing at a fixed origin O observes an insect taking off from a point A on level horizontal ground. The position vector of the insect, t seconds after taking off is given by

$$\underset{\sim}{r} = (t+1)\underset{\sim}{i} + \left(2t + \frac{1}{2}\right)\underset{\sim}{j} + 2t\underset{\sim}{k}.$$

All distances are in metres and the coordinate axes x, y, z are oriented due east, due north and vertically upwards, respectively.

(i) Find the bearing of the insect's flight path, correct to the nearest degree. **2**

(ii) Calculate the angle between the flight path and the horizontal ground, to the nearest degree. **1**

A bird is waiting on the roof top of a garden shed at a point B with coordinates $\left(5, \frac{9}{2}, 3\right)$.

(iii) Let $D(x, y, z)$ be the point on the insect's path which is the shortest distance between the insect's path and the bird at the point B.
By first finding the projection of $\overrightarrow{AB}$ onto the insect's flight path, calculate the shortest distance between the bird and the insect's path. **3**

Question 16 (14 Marks) Start on the relevant page of your answer booklet

(a) Let $I_n = \int_0^{\lambda} x^n e^{-x} dx$ for positive integers $n > 0$.

(i) Show that $I_n = n\, I_{n-1} - e^{-\lambda}\lambda^n$ for $n \geq 1$. **3**

(ii) If $J_n = \lim_{\lambda \to \infty} I_n$, show that $J_n = n\, J_{n-1}$ for $n \geq 1$. **1**

(iii) Deduce that $J_n = n!$ for $n \geq 1$. **2**

(b) Given p and q are positive real numbers:

(i) Prove that $\frac{1}{2}(p+q) \geq \sqrt{pq}.$ **1**

(ii) Hence, deduce that $\sqrt{p} \leq \frac{1}{2}\left(\frac{p}{\sqrt{q}} + \sqrt{q}\right).$ **1**

(c) Given $a_1, a_2, a_3, ..., a_n$ and $b_1, b_2, b_3, ..., b_n$ are positive real numbers, where $0 < b_1 \leq b_2 \leq b_3 \leq ... \leq b_n$,

and $A_n = a_1 + a_2 + a_3 + ... + a_n$, $B_n = b_1 + b_2 + b_3 + ... + b_n$, where $A_r \leq B_r$, for $r = 1, 2, 3, ..., n$.

(i) Prove, by mathematical induction for all positive integers n, that **3**

$$\frac{1}{\sqrt{b_n}} B_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right) B_{n-1} + \left(\frac{1}{\sqrt{b_{n-2}}} - \frac{1}{\sqrt{b_{n-1}}}\right) B_{n-2} + ... + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1$$
$$= \sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3} + ... + \sqrt{b_n}.$$

(ii) Hence, given that

$$\frac{a_1}{\sqrt{b_1}} + \frac{a_2}{\sqrt{b_2}} + \frac{a_3}{\sqrt{b_3}} + ... + \frac{a_n}{\sqrt{b_n}}$$
$$= \frac{1}{\sqrt{b_n}} A_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right) A_{n-1} + \left(\frac{1}{\sqrt{b_{n-2}}} - \frac{1}{\sqrt{b_{n-1}}}\right) A_{n-2} + ... + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) A_1,$$

show that $\sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}} \leq \sum_{r=1}^{n} \sqrt{b_r}.$ **1**

(iii) Using the result from Question 16 part (b), or otherwise, deduce that **2**

$$\sum_{r=1}^{n} \sqrt{a_r} \leq \sum_{r=1}^{n} \sqrt{b_r}.$$

End of Paper

MATHEMATICS EXTENSION 2 2023 TRIAL SOLUTIONS

MULTIPLE CHOICE SOLUTIONS

1. $\vec{OA} + \vec{AB} = \vec{OB}$

$\vec{OA} = \vec{OB} - \vec{AB}$

$= \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$\vec{OA} = \sqrt{2^2 + 2^2 + 1^2}$

$= \sqrt{9}$

$= 3$

$\therefore$ B

2. Only one is false.

Use a counterexample to prove B is false.

Let $x = -3$, $y = 2$

$x^2 - y^2 = 9 - 4$ $\qquad$ $x - y = -3 - 2$

$= 5$ $\qquad$ $= -5$

> 0 $\qquad$ < 0 which is false.

$\therefore$ D

3. $\int 3\sqrt{x} \ln x$

$= 2x^{3/2} \ln x - \int 2x^{\frac{1}{2}} \, dx$

$= 2x^{3/2} \ln x - 2 \times \frac{2}{3} x^{3/2} + C$

$= 2x\sqrt{x}\left(\ln x - \frac{2}{3}\right) + C$

$\therefore$ A

4. $\int \frac{1}{x^2 + 6x + 25} \, dx = \int \frac{dx}{(x+3)^2 + 4^2}$

$= \frac{1}{4} \tan^{-1}\left(\frac{x+3}{4}\right) + C$

$\therefore$ A

5. $a = v\frac{dv}{dx}$ $\qquad$ $v = f(x)$

$= f(x) f'(x)$ $\qquad$ $\frac{dv}{dx} = f'(x)$

$\therefore$ D

6. $$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}|\,|\overrightarrow{PR}|}$$

$$= \frac{\begin{pmatrix}6-4\\3-2\\-1+3\end{pmatrix}\cdot\begin{pmatrix}5-4\\1-2\\0+3\end{pmatrix}}{\left|\begin{pmatrix}2\\1\\2\end{pmatrix}\right|\left|\begin{pmatrix}1\\-1\\3\end{pmatrix}\right|}$$

$$= \frac{\begin{pmatrix}2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}}{\sqrt{9}\times\sqrt{11}}$$

$$= \frac{2-1+6}{3\sqrt{11}}$$

$$\cos\theta = \frac{7}{3\sqrt{11}}$$

$$\theta = \cos^{-1}\frac{7}{3\sqrt{11}}$$

$\theta = 45^\circ$ (nearest degree)

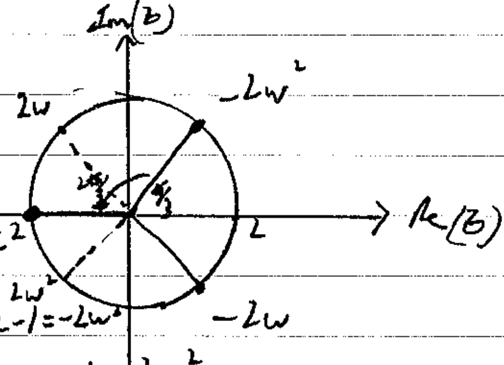
$\therefore$ C

7. $(z-1)^3 = -8$

Consider $z^3 = 8$ and $z^3 = -8$

Roots of $z^3 = 8$ are $2, 2w, 2w^2$

Roots of $z^3 = -8$ are $-2, -2w, -2w^2$



$\therefore z-1 = -2 \quad z-1 = -2w \quad z-1 = -2w^2$

$z = -1 \quad z = 1-2w \quad z = 1-2w^2$

$\therefore$ C

8. Forces on m_1:

$F = ma$

$m_1 a = m_1 g - T$

$T = m_1 g - m_1 a$ (1)

Forces on m_2:

$F = ma$

$m_2 a = T - m_2 g$ (2)

Substitute (1) in (2)

$$m_2 a = m_1 g - m_1 a - m_2 g$$

$$a(m_1 + m_2) = g(m_1 - m_2)$$

$$\therefore a = \frac{g(m_1 - m_2)}{m_1 + m_2}$$

$\therefore$ B

9. $\overline{\sin x + i\cos 2x} = \cos x - i\sin 2x$

$\sin x - i\cos 2x = \cos x - i\sin 2x$

$\sin x = \cos x$ <u>and</u> $\cos 2x = \sin 2x$ (equating real and imaginary parts)

$\tan x = 1$ and $\tan 2x = 1$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

But if $\tan x = 1$ $\tan 2x$ is undefined (denominator is 0)

$\therefore$ No value of x $\therefore$ A

10. $z\bar{z} + a(z+\bar{z}) + a = 0$

$x^2+y^2+2ax + a = 0$

$(x+a)^2 + y^2 = a^2 - a$ $\quad r^2 = a^2 - a$

Also $a^2 - 7a + 10 = 0$

$(a-5)(a-2) = 0$

$\therefore a = 2$ or 5 $\quad$ (1)

Since area of a circle $= \pi r^2$

$r^2 = 2$

$a^2 - a = 2$

$a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

$a = -1$ or 2. $\quad$ (2)

Combining (1) and (2), $a = 2$.

For $z\bar{z} + 2a(z+\bar{z}) + a = 0$

$x^2+y^2+4ax + a = 0$

$(x+2a)^2 + y^2 = 4a^2 - a$

Area $= \pi(4\times 2^2 - 2)$

$= 14\pi$ square units

$\therefore$ C

11 a) $\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx$

$= \int 1 - \frac{1}{x^2+1}\,dx$

$= x - \tan^{-1}x + C$

(2) Correct answer
(1) obtains $\int 1 - \frac{1}{x^2+1}\,dx$

b) i) $z = \frac{1-7i}{3+4i}\,\frac{3-4i}{3-4i}$

$= \frac{3-25i+28i^2}{9+16}$

$= \frac{-25-25i}{25}$

$= -1-i$

(1) Correct answer

ii) $\mathrm{Arg}(-1-i)^2 = 2\,\mathrm{Arg}(-1-i)$

$= 2 \times \frac{-3\pi}{4}$

$= -\frac{6\pi}{4}$

$= \frac{\pi}{2}$

OR $\mathrm{Arg}(-1-i)^2 = \mathrm{Arg}(2i)$

$= \frac{\pi}{2}$

(2) Correct answer
(1) Obtains $\frac{-3\pi}{4}$ or $2i$

Note: must be principal argument

c) i) $\frac{5x^2-4x-9}{(x-2)(x^2-3)} \equiv \frac{a}{x-2} + \frac{bx+c}{x^2-3}$

$5x^2-4x-9 \equiv a(x^2-3) + (x-2)(bx+c)$

Let $x=2$ $\quad 3 = a$

Comparing coefficients of x^2: $5 = a+b$

$5 = 3+b$

$b = 2$

x^0: $-9 = -3a-2c$

$2c = 0$

$c = 0$

$\therefore a=3, b=2, c=0$

(2) correct values of a, b, c
(1) finds a, b or c

11 c ii) $\int_3^4 \frac{5x^2-4x-9}{(x-2)(x^2-3)}\,dx = \int_3^4 \left(\frac{3}{x-2} + \frac{2x}{x^2-3}\right) dx$

$= \Big[3\ln|x-2| + \ln|x^2-3|\Big]_3^4$

$= (3\ln 2 + \ln 13) - (3\ln 1 + \ln 6)$

$= \ln 8 + \ln 13 - \ln 6$

$= \ln \frac{8\times 13}{6}$

$= \ln \frac{52}{3}$

(2) correct solution
(1) correctly integrates

d i) $z_1 = e^{i\frac{\pi}{3}}$, $z_2 = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$z_1 z_2 = \sqrt{2}\, e^{i\left(\frac{\pi}{3}+\frac{\pi}{4}\right)}$

$= \sqrt{2}\, e^{i\frac{7\pi}{12}}$

(2) correct solution
(1) expresses z_1 or z_2 in exponential form

ii) $(z_1 z_2)^n = \left(\sqrt{2}\, e^{i\frac{7\pi}{12}}\right)^n$

$= \sqrt{2}^{\,n}\, e^{i\frac{7\pi n}{12}}$

To be purely imaginary

$\frac{7\pi n}{12} = \frac{k\pi}{2}$ where k is an integer

$n = \frac{6}{7}$

k will be an integer whenever n is a multiple of $\frac{6}{7}$

$\therefore$ Smallest value of n is $\frac{6}{7}$

(2) correct solution
(1) identifies condition for $z_1 z_2$ to be purely imaginary
(Note students who used De Moivre's Theorem must have an integer value)

e) i) If $x<0$, then $x^5+4x^3+5x < 8x^4+12x^2+17$ (1) correct answer

ii) If $x<0$ then $x^5+4x^3+5x<0$

If $x<0$ then $8x^4+12x^2+17>0$

$\therefore 8x^4+12x^2+17 > x^5+4x^3+5x$

(2) correct solution
(1) Justifies $x^5+4x^3+5x<0$ or $8x^4+12x^2+17>0$

12 a) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{dx}{1+\sin x}$

Let $t = \tan\frac{x}{2}$

$dx = \frac{2dt}{1+t^2}$

When $x = \frac{2\pi}{3}$, $t = \sqrt{3}$

$x = \frac{\pi}{3}$, $t = \frac{1}{\sqrt{3}}$

(3) correct solution

(2) correctly integrates in terms of t

(1) expresses in terms of t

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{\left(1+\frac{2t}{1+t^2}\right)} \frac{2dt}{(1+t^2)}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{2dt}{t^2+1+2t}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{2dt}{(t+1)^2}$$

$$= 2\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} (t+1)^{-2}\, dt$$

$$= 2\left[\frac{(t+1)^{-1}}{-1}\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= -2\left(\frac{1}{\sqrt{3}+1} - \frac{1}{\frac{1}{\sqrt{3}}+1}\right)$$

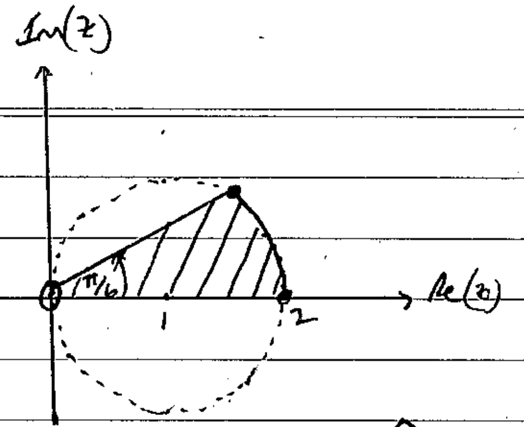
$$= -2\left(\frac{1}{\sqrt{3}+1} - \frac{\sqrt{3}}{1+\sqrt{3}}\right)$$

$$= -2\left(\frac{1-\sqrt{3}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}\right)$$

$$= -2\frac{(4-2\sqrt{3})}{-2}$$

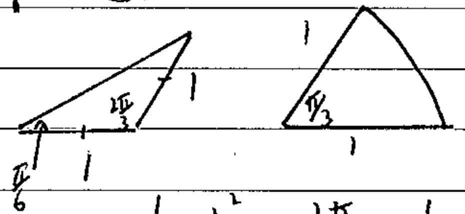
$$= 4-2\sqrt{3}$$

12 b i)



(2) correct diagram
(1) correct region with incorrect boundaries
OR (1) correct boundaries with incorrect region

ii)



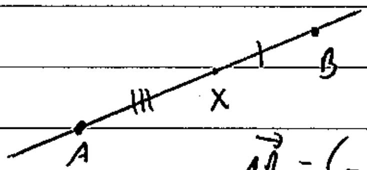
$\text{Area} = \frac{1}{2} \times 1^2 \times \sin\frac{2\pi}{3} + \frac{1}{2} \times 1^2 \times \frac{\pi}{3}$

$= \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{\pi}{3}$

$= \frac{\sqrt{3}}{4} + \frac{\pi}{6}$ units^2

(2) correct answer
(1) identifies triangle and sector and correctly finds one area

12 c)



$\vec{AB} = (-1, 2, -8)$

$\vec{OX} = \vec{OA} + \frac{3}{4}\vec{AB}$

$= (2, -6, 5) + \frac{3}{4}(-1, 2, -8)$

$= (2, -6, 5) + \left(-\frac{3}{4}, \frac{3}{2}, -6\right)$

$= \left(\frac{5}{4}, -\frac{9}{2}, -1\right)$

(2) correct solution
(1) finds $\vec{AB}$ or equivalent progress

12 d)

$x = \frac{2 \pm \sqrt{4-16}}{2}$

$= \frac{2 \pm \sqrt{-12}}{2}$

$= \frac{2 \pm 2\sqrt{3}i}{2}$

$x = 1 \pm \sqrt{3}i$

let $\alpha = 1 + \sqrt{3}i$ and $\beta = 1 - \sqrt{3}i$

$\alpha = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$ $\quad \beta = 2\operatorname{cis}\left(\frac{-\pi}{3}\right)$

$\alpha^n + \beta^n = 2^n\left(\operatorname{cis}\frac{\pi}{3}\right)^n + 2^n\left(\operatorname{cis}\left(\frac{-\pi}{3}\right)\right)^n$

$= 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\left(\frac{-n\pi}{3}\right) + i\sin\left(\frac{-n\pi}{3}\right)\right)$ By De Moivre's Theorem

$= 2^n\left(2\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$ $\cos\theta$ is even, $\sin\theta$ is odd

$= 2^{n+1}\cos\frac{n\pi}{3}$

(3) correct solution
(2) Applies De Moivre's theorem to find α^n or β^n
(1) Finds α and β

12 e) $z_1 - z_2 + z_3 - z_4 = 0$

$z_1 - z_2 = z_4 - z_3$

Since $z_1 - z_2 = z_4 - z_3$, opposite sides are parallel

$|z_1 - z_2| = |z_4 - z_3|$, opposite sides equal

$\therefore$ ABCD is a parallelogram (one pair of opposite sides are equal and parallel)

(2) correct solution
(1) Identifies sides are equal or parallel

13 a) $\int \frac{e^x dx}{e^{2x} - 1} dx$

Let $u = e^x$

$du = e^x dx$

$= \int \frac{du}{u^2 - 1}$

$= \int \frac{du}{(u+1)(u-1)}$

$= \int \frac{-1}{2(u+1)} + \frac{1}{2(u-1)} du$

$= \frac{1}{2}\left(\ln|u-1| - \ln|u+1|\right) + C$

$= \frac{1}{2} \ln\left|\frac{u-1}{u+1}\right| + C$

$= \frac{1}{2} \ln\left|\frac{e^x - 1}{e^x + 1}\right| + C$

(3) correct solution
(2) correct primitive in terms of u
(1) correct integral in terms of u

b i) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(6 + 4x - 2x^2\right)$

$= 4 - 4x$

$= -4(x-1)$

Since $\ddot{x}$ is of the form $\ddot{x} = -n^2(x-a)$ where $n = 2, a = 1$, it moves in SHM

(2) correct solution
(1) uses $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to find acceleration

ii) For amplitude: Let $v = 0$, $0 = 3 + 2x - x^2$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = -1, 3$

$\therefore$ Amplitude $= 2$

(3) correct solution
(2) two correct values
(1) one correct value

centre $= 1$ from $\ddot{x} = -4(x-1)$

period $= \frac{2\pi}{2}$

$= \pi$

13 b iii) since initially stationary at RHS endpoint

$x = 1 + 2\cos(2t)$

when $x = 2$ $\quad 1 = 2\cos(2t)$

$\cos 2t = \frac{1}{2}$

$2t = \frac{\pi}{3}$

$t = \frac{\pi}{6}$

$\therefore$ It takes $\frac{\pi}{6}$ second to travel 1 metre

Now when $x = 2$, $v^2 = 12$

$v = \pm 2\sqrt{3}$

since moving left at first time it passes $x = 1$, velocity $= -2\sqrt{3}$

[or $\dot{x} = -2 \times 2\sin(2t)$

when $t = \frac{\pi}{6}$ $\dot{x} = -4\sin\frac{\pi}{3}$

$= -\frac{4\sqrt{3}}{2}$

$= -2\sqrt{3}$ m/s.

$\therefore$ Takes $\frac{\pi}{6}$ seconds and has velocity of $-2\sqrt{3}$ m/s

(3) correct solution

(2) obtains correct displacement equation and velocity

or (2) finds time to travel 1 metre

(1) finds displacement equation OR velocity

13 c) Equation of BE

$\vec{DE} = (2,1,0) - (5,1,4)$
$= (-3,0,-4)$

$\ell = (2,1,0) + \lambda(-3,0,-4)$
$\therefore (x,y,z) = (-3\lambda+2, 1, -4\lambda)$ $\quad (*)$

$\vec{CB} = (x,y,z) - (3,7,12)$
$= (x-3, y-7, z-12)$

$\vec{CD} = (5,1,4) - (3,7,12)$
$= (2,-6,-8)$

Since $\vec{CB} \perp \vec{CD}$

$(x-3, y-7, z-12) \cdot (2,-6,-8) = 0$
$2x - 6 - 6y + 42 - 8z + 96 = 0$
$x - 3y - 4z + 66 = 0$

From $(*)$ above

$-3\lambda + 2 - 3 - 4 \times -4\lambda + 66 = 0$
$-3\lambda + 16\lambda = -65$
$13\lambda = -65$
$\lambda = -5$

$\therefore B$ is $(17, 1, 20)$

$\vec{OA} = \vec{OB} + \vec{BA}$
$= (17,1,20) + (2,-6,-8)$ since $\vec{BA} = \vec{CD}$ (opposite sides of a rectangle are equal and parallel)
$= (19,-5,12)$
$\therefore A$ is $(19,-5,12)$.

(4) Correct solution

(3) finds equation of the line BDE and dot product of $\vec{CB}$ and $\vec{CD}$

(2) finds equation of the line DE and finds $\vec{CB}$ or $\vec{CD}$

(1) finds equation of the line DE or finds $\vec{CB}$ or $\vec{CD}$ or dot product

14 a) Suppose, by contradiction, that positive integers p and q do exist such that $4p^2-q^2=25$

Then $(2p+q)(2p-q) = 25$

Factors of 25 are 1, 5, 25

Since $q \neq 0$ (q is a positive integer) then factors must be 1 and 25 as they are distinct.

$2p-q=1$ ① and $2p+q=25$ ②

② − ① $2q=24$

$q=12$

If $q=12$, $2p+q=25$

$2p=13$

$p=6\frac{1}{2}$

∴ A contradiction exists since p is a positive integer.

∴ There are no positive integers p and q such that $4p^2-q^2=25$

④ correct solution

③ determines $p=6\frac{1}{2}$ and $q=12$ are the only possibilities

② equates $2p+q=25$ and $2p-q=1$

① obtains $(2p+q)(2p-q)=25$

(Note: student must specify $2p-q<2p+q$ OR $2p-q=1$ and $2p+q=25$

14b) i) $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ AM/GM Inequality

$a+b+c \geq 3\sqrt[3]{abc}$

$(a+b+c)^3 \geq 27abc$

② correct solution

① Applies AM/GM Inequality to the 3 terms

ii) let $a \to \frac{a}{b^4}$, $b \to \frac{b}{c^4}$, $c \to \frac{c}{a^4}$

$$\frac{\frac{a}{b^4}+\frac{b}{c^4}+\frac{c}{a^4}}{3} \geq \sqrt[3]{\frac{abc}{a^4b^4c^4}}$$

$$\frac{a}{b^4}+\frac{b}{c^4}+\frac{c}{a^4} \geq 3\sqrt[3]{\frac{1}{(abc)^4}} \quad (\text{since } abc=1)$$

$$a^4b^4c^4\left(\frac{a}{b^4}+\frac{b}{c^4}+\frac{c}{a^4}\right) \geq 3\sqrt[3]{\frac{1}{1}}(abc)^4$$

$$a^5c^4+b^5a^4+c^5b^4 \geq 3$$

③ correct solution

② obtains $\frac{a}{b^4}+\frac{b}{c^4}+\frac{c}{a^4} \geq 1$

① Identifies correct substitution or uses $abc=$

14c) i) $z^n = 1$ has roots $w_1, w_2, \dots w_n$

Sum of roots from $z^n - 1 = 0$ is

$$w_1 + w_2 + \dots w_n = \frac{-0}{1}$$
$$= 0$$

① correct proof

ii) $$|PA_k|^2 = |z - w_k|^2$$
$$= (z - w_k)(\overline{z - w_k})$$
$$= (z - w_k)(\bar{z} - \bar{w}_k)$$

① correct proof

iii) $$\sum_{k=1}^{n} |PA_k|^2 = (z - w_1)(\bar{z} - \bar{w}_1) + (z - w_2)(\bar{z} - \bar{w}_2) + \dots + (z - w_n)(\bar{z} - \bar{w}_n)$$
$$= (z\bar{z} - z\bar{w}_1 - \bar{z}w_1 + w_1\bar{w}_1) + (z\bar{z} - z\bar{w}_2 - \bar{z}w_2 + w_2\bar{w}_2) + \dots + (z\bar{z} - z\bar{w}_n - \bar{z}w_n + w_n\bar{w}_n)$$
$$= (|z|^2 - z\bar{w}_1 - \bar{z}w_1 + |w_1|^2) + (|z|^2 - z\bar{w}_2 - \bar{z}w_2 + |w_2|^2) + \dots + (|z|^2 - z\bar{w}_n - \bar{z}w_n + |w_n|^2)$$
$$= \underbrace{(1+1) + (1+1) + \dots (1+1)}_{n \text{ times}} - z(\bar{w}_1 + \bar{w}_2 + \dots + \bar{w}_n) - \bar{z}(w_1 + w_2 + \dots + w_n)$$

(since $|z| = 1$ & $|w_k| = 1$)

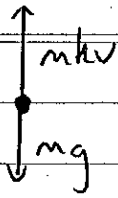
$$= 2n - z(\overline{w_1 + w_2 + \dots + w_n}) - \bar{z} \times 0$$
$$= 2n - z \times 0$$
$$= 2n$$

③ correct proof

② applies $z\bar{z} = |z|^2$ to simplify the expression

① applies the identity from (ii) and attempts to expand binomial products

15 a) i)



① correct solution

$$ma = mg - mkv$$
$$a = g - kv$$

ii) For terminal velocity $a = 0$

① correct solution

$$0 = g - kv$$
$$kv = g$$
$$v = \frac{g}{k}$$

$\therefore$ Terminal velocity is $\frac{g}{k}$ ms^{-1}

iii)

④ correct solution
③ correctly integrates with limits
② integrates velocity correctly
① separates variables to form two appropriate integrals

$$v\frac{dv}{dx} = g - kv$$
$$\int_0^{V_0} \frac{v\,dv}{g-kv} = \int_0^H dx$$
$$\frac{-1}{k}\int \frac{-kv\,dv}{g-kv} = [x]_0^H$$
$$H - 0 = -\frac{1}{k}\int_0^{V_0} \frac{g-kv-g}{g-kv}\,dv$$
$$H = -\frac{1}{k}\int_0^{V_0} 1 - \frac{g}{g-kv}\,dv$$
$$H = -\frac{1}{k}\left[V\right]_0^{V_0} + \frac{g}{k}\left(-\frac{1}{k}\right)\int_0 \frac{-k\,dv}{g-kv}$$
$$H = -\frac{1}{k}(V_0 - 0) - \frac{g}{k^2}\left[\ln|g-kv|\right]_0^{V_0}$$
$$H = -\frac{V_0}{k} - \frac{g}{k^2}\left[\ln|g-kV_0| - \ln g\right]$$
$$H = \frac{-V_0}{k} - \frac{g}{k^2}\ln\left|\frac{g-kV_0}{g}\right|$$
$$\frac{g}{k^2}\ln\left|\frac{g-kV_0}{g}\right| + \frac{V_0}{k} + H = 0$$

$\times \frac{k^2}{g}$:

$$\ln\left|1 - \frac{kV_0}{g}\right| + \frac{kV_0}{g} + \frac{Hk^2}{g} = 0 \text{ as required}$$

15 a) iv) $\frac{dv}{dt} = g - kv$

③ correct solution
② correctly integrates from acceleration
① correct integrand

$$\int_0^{V_0} \frac{dv}{g-kv} = \int_0^T dt$$

$$\left[t\right]_0^T = -\frac{1}{k}\int_0^{V_0} \frac{-k\,dv}{g-kv}$$

$$T - 0 = \frac{-1}{k}\left[\ln|g-kv|\right]_0^{V_0}$$

$$T = -\frac{1}{k}\left[\ln|g-kV_0| - \ln g\right]$$

$$T = -\frac{1}{k}\ln\left|\frac{g-kV_0}{g}\right|$$

$$T = \frac{1}{k}\ln\left(\frac{g}{g-kV_0}\right)$$

$$-kT = \ln\left(\frac{g-kV_0}{g}\right)$$

$$-kT = \ln\left(1 - \frac{kV_0}{g}\right) = -\frac{1}{k}\ln\left(\frac{g-kV_0}{g}\right)$$

$$T = \frac{1}{k}\ln\left(\frac{g}{g-kV_0}\right) \text{ as required}$$

v) $$-kT = \frac{-kV_0}{g} - \frac{k^2 H}{g}$$

① Correct proof

$$\left(\times\frac{g}{k}\right): \quad -gT = -V_0 - kH$$

$$\therefore V_0 = Tg - kH$$

vi) $V_0 < V_T$ where V_T = terminal velocity

① correct proof

At terminal velocity,

from ii) $v = \frac{g}{k}$

$$\therefore V_0 < \frac{g}{k}$$

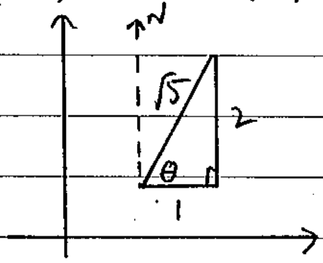
$$Tg - kH < \frac{g}{k} \quad \text{from (iii)}$$

$$Tg < \frac{g}{k} + kH$$

$$T < \frac{1}{k} + \frac{kH}{g}$$

15 d i) $L = \left(1, \frac{1}{2}, 0\right) + t\left(1, 2, 2\right)$

(2) correct solution
(1) Identifies direction vector as $(1, 2, 2)$ OR correctly obtains $63°$

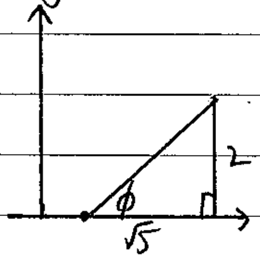


$\tan\theta = 2$

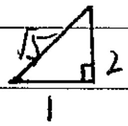
$\theta = 63°$

$\therefore$ Bearing $= 027°T$

d ii)



Top view from (i)

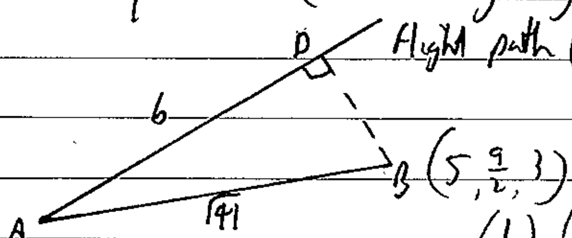


(1) correct answer

$\tan\phi = \frac{2}{\sqrt{5}}$

$\phi = 42°$ (nearest degree)

d iii)



(3) correct solution
(2) finds projection of AB onto flight path
(1) finds $\overrightarrow{AB}$

Projection $\overrightarrow{AB}$ onto flight path $= \dfrac{\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}5-1\\ \frac{9}{2}-\frac{1}{2}\\ 3-0\end{pmatrix}}{\left(\sqrt{2^2+2^2+1}\right)^2}\begin{pmatrix}1\\2\\2\end{pmatrix}$

$= \dfrac{\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}4\\4\\3\end{pmatrix}}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}$

$= 2\begin{pmatrix}1\\2\\2\end{pmatrix}$

$= \begin{pmatrix}2\\4\\4\end{pmatrix}$

$|\text{Projection}| = \sqrt{4+16+16}$

$= 6$

$|AB| = |16+16+9|$

$= \sqrt{41}$

Perpendicular distance / shortest distance $= \sqrt{41^2 - 6^2}$

$= \sqrt{5}$

$\therefore$ Shortest distance $= \sqrt{5}$ m

16 a) i) $I_n = \int_0^{\lambda} x^n e^{-x}\,dx$ $\quad u = x^n \quad v' = e^{-x}$

$u' = nx^{n-1} \quad v = -e^{-x}$

(3) correct proof
(2) recognises I_{n-1}
(1) applies integration by parts

$$= \left[-x^n e^{-x}\right]_0^{\lambda} + \int_0^{\lambda} n x^{n-1} e^{-x}\,dx$$

$$= -\lambda^n e^{-\lambda} - 0 + n\int_0^{\lambda} x^{n-1} e^{-x}\,dx$$

$$= nI_{n-1} - \lambda^n e^{-\lambda}$$

ii) $J_n = \lim_{\lambda \to \infty} I_n$

(1) correct solution

$$= \lim_{\lambda \to \infty}\left(-\lambda^n e^{-\lambda}\right) + \lim_{n \to \infty} nI_{n-1}$$

$$= 0 + \lim_{n \to \infty} nI_{n-1}$$

$$J_n = nJ_{n-1}$$

iii) $J_n = nJ_{n-1}$

(2) correct proof
(1) evaluates J_0 or expresses J_n in terms of J_0

$$= n(n-1)J_{n-2}$$

$$= n(n-1)\ldots J_0$$

$$J_0 = \lim_{n \to \infty} I_0$$

$$= \lim_{n \to \infty} \int_0^{\lambda} e^{-x}\,dx$$

$$= \lim_{n \to \infty} \left[-e^{-x}\right]_0^{\lambda}$$

$$= \lim_{n \to \infty}\left(-e^{-\lambda} + e^0\right)$$

$$= 0 + 1$$

$$= 1$$

$$\therefore J_n = n(n-1)\ldots 1$$

$$= n!$$

16 b (i) Consider $(\sqrt{p}-\sqrt{q})^2 \geq 0$ ① correct solution

$$p - 2\sqrt{pq} + q \geq 0$$
$$p+q \geq 2\sqrt{pq}$$
$$\frac{1}{2}(p+q) \geq \sqrt{pq}$$

ii) Dividing both sides of (i) by $\sqrt{q}$: ① correct solution

$$\sqrt{p} \leq \frac{1}{2}\left(\frac{p}{\sqrt{q}} + \frac{q}{\sqrt{q}}\right)$$

$\therefore \sqrt{p} \leq \frac{1}{2}\left(\frac{p}{\sqrt{q}} + \sqrt{q}\right)$ as required

16 c (i) For $n=1$

③ Correct proof

② proves inductive step OR proves base case and uses assumption

① proves base case OR uses assumption to attempt to prove inductive step

$$RHS = \sqrt{b_1}$$
$$LHS = \frac{1}{\sqrt{b_1}} B_1$$
$$= \frac{1}{\sqrt{b_1}} b_1$$
$$= \sqrt{b_1}$$
$$= LHS$$

$\therefore$ True for $n=1$

Assume true for $n=k$

$$\frac{1}{\sqrt{b_k}} B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right)B_{k-1} + \dots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right)B_1 = \sqrt{b_1} + \sqrt{b_2} + \dots + \sqrt{b_n}$$

For $n=k+1$ we wish to prove

$$\frac{1}{\sqrt{b_{k+1}}} B_{k+1} + \left(\frac{1}{\sqrt{b_k}} - \frac{1}{\sqrt{b_{k+1}}}\right)B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right)B_{k-1} + \dots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right)B_1 = \sqrt{b_1} + \sqrt{b_2} + \dots + \sqrt{b_k} + \sqrt{b_{k+1}}$$

$$\frac{B_{k+1}}{\sqrt{b_{k+1}}} + \frac{B_k}{\sqrt{b_k}} - \frac{B_k}{\sqrt{b_{k+1}}} + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right)B_{k-1} + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right)B_1$$

$$= \frac{B_{k+1}}{\sqrt{b_{k+1}}} - \frac{B_k}{\sqrt{b_{k+1}}} + \sqrt{b_1} + \sqrt{b_2} + \dots + \sqrt{b_k}$$

$$= \sqrt{b_1} + \sqrt{b_2} + \dots \sqrt{b_k} + \frac{1}{\sqrt{b_{k+1}}}\left(B_{k+1} - B_k\right)$$

$$= \sqrt{b_1} + \sqrt{b_2} + \ldots + \sqrt{b_k} + \frac{1}{\sqrt{b_{k+1}}}\left(b_1 + b_2 + \ldots + b_k + b_{k+1} - b_1 + b_2 + b_3 + \ldots + b_k\right)$$

$$= \sqrt{b_1} + \sqrt{b_2} + \ldots + \sqrt{b_k} + \frac{b_{k+1}}{\sqrt{b_{k+1}}}$$

$$= \sqrt{b_1} + \sqrt{b_2} + \ldots + \sqrt{b_k} + \sqrt{b_{k+1}}$$ as required.

$\therefore$ If true for $n=k$, the result is true for $n=k+1$. But it is true for $n=1$, therefore it is true by induction for all positive integers $n \geq 1$.

ii) since $A_r \leq B_r$ for $r = 1, 2, \ldots, n$ ① correct proof

$$\sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}} \leq \frac{B_n}{\sqrt{b_n}} + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right)B_{n-1} + \ldots \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right)B_1$$

$$= \sum_{r=1}^{n} \sqrt{b_r}$$ from (i)

$$\therefore \sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}} \leq \sum_{r=1}^{n} \sqrt{b_r}$$

iii) From part (b) ② correct proof ① applies result from part (b)

$$\sqrt{a_r} \leq \frac{1}{2}\left(\frac{a_r}{\sqrt{b_r}} + \sqrt{b_r}\right)$$

$$\sum_{r=1}^{n} \sqrt{a_r} \leq \frac{1}{2}\left(\sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}} + \sum_{r=1}^{n} \sqrt{b_r}\right)$$

$$\sum_{r=1}^{n} \sqrt{a_r} \leq \frac{1}{2}\left(\sum_{r=1}^{n} \sqrt{b_r} + \sum_{r=1}^{n} \sqrt{b_r}\right)$$ from (ii)

$$\sum_{r=1}^{n} \sqrt{a_r} \leq \frac{1}{2}\left(2\sum_{r=1}^{n} \sqrt{b_r}\right)$$

$$\therefore \sum_{r=1}^{n} \sqrt{a_r} \leq \sum_{r=1}^{n} \sqrt{b_r}$$ as required.